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Dynamical Symmetries in Relativistic Kinetic Theory

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This paper is part of a program investigating symmetries that are defined at a physical or observational level rather than purely geometrically. Here we generalize previous work on dynamical "matter" symmetries of relativistic gases. If the matter symmetry vector is surface-forming with the dynamical Liouville vector, then Einstein's equations reduce it to a Killing symmetry of the metric. We show that this conclusion is unaltered if the gas particles are subject to a nongravitational force (including the electromagnetic force on charged particles) or if the gravitational field obeys higher-order field equations. In the Brans–Dicke theory, the matter symmetry reduces to a homothetic symmetry of the metric. This is also the case for a generalized conformal symmetry in Einstein's theory. We consider the problem of relaxing the surface-forming assumption in an attempt to determine whether there are dynamical symmetries that do not necessarily reduce to geometrical symmetries of the metric.

1. INTRODUCTION

Much work has been done on symmetries in fluid spacetimes (see Kramer *et al.*, 1980; Coley and Tupper, 1990; and references therein). The approach in most cases has been to assume, sometimes without clear physical or observational reasons, a geometrical symmetry of spacetime and then to consider the consequences for the fluid properties. Similarly, most work on symmetries in relativistic kinetic theory starts from a geometrical symmetry of spacetime and investigates the consequences for the distribution function $f(x, p)$ (see Maartens and Maharaj and references therein). This paper is motivated by a desire to define a fundamental symmetry at a *physically observational or dynamical* level rather than purely geometrically. Previous attempts include the postulate of uniform thermal histories of Bonnor and Ellis (1986), which aimed to give an observational

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definition of homogeneity in fluid spacetimes; dynamical path symmetries of particle motion (see Maartens and Taylor, 1993, and references therein); and the dynamical matter symmetries defined in relativistic kinetic theory by Berezdivin and Sachs (1973). In this paper, we take the view that any fundamental concept of dynamical symmetry should have a foundation in the microscopic model of relativistic kinetic theory. Therefore, we aim to generalize the work of Berezdivin and Sachs (1973) on dynamical matter symmetries. A previous paper (Maartens and Taylor, 1993) dealt with generalizing work on dynamical path symmetries.

There is an important distinction between the dynamical matter symmetries of Berezdivin and Sachs and dynamical path symmetries of particle motion (Maartens and Taylor, 1993). The latter symmetries are defined by an invariance of phase space orbits. It is not clear whether they have any observable consequence, since they map orbits in phase space into each other. Dynamical matter symmetries, on the other hand, may be defined directly in terms of measurements of the particle distribution function (Berezdivin and Sachs, 1973). It is shown in Maartens and Taylor (1993) that matter and path symmetries are equivalent only if they both arise from an underlying homothetic symmetry of the spacetime metric.

Our investigation of matter symmetries aims to clarify the relation between these dynamical symmetries and geometrical symmetries. Although matter symmetries in relativistic kinetic theory are an important concept in a dynamical approach to symmetries, they have received little attention. We have recently situated matter symmetries within a broader class of vector fields on the tangent bundle (Berezdivin and Sachs, 1973). Oliver and Davis (1979) considered some mathematical properties of matter symmetries in the context of classifying geometrical symmetries. However, as far as we are aware, no work has been done to extend the original kinetic theory results of Berezdivin and Sachs. We aim to do so in this paper.

Berezdivin and Sachs were unable to find a solution to the general case and had to assume that the matter symmetry vector field and the Liouville vector field (geodesic spray) were 2-surface-forming in phase space. With this assumption, they showed that Einstein's field equations imply that the dynamical matter symmetry necessarily arises from an isometry of the spacetime. In other words, the field equations transmit the dynamical symmetry directly to the geometry. We try to generalize their result in various directions: (a) considering the effect of nongravitational forces; (b) considering alternative field equations; (c) generalizing the definition of matter symmetry to a conformal matter symmetry; and (d) considering matter symmetry vector fields that form 3-dimensional integral surfaces with the Liouville vector field.

Section 2 contains a brief summary of relativistic kinetic theory for a collision-free gas. Section 3 is a brief review of lifted vector fields and transformations on the tangent bundle, which includes the Berezdivin and Sachs matter symmetry as a particular case. Section 4 outlines the results of Berezdivin and Sachs and provides an alternative derivation of their main result. We believe that our derivation is more direct and clear. Also, it may be extended to the more general cases, unlike the Berezdivin and Sachs methods. Section 5 details the extensions and generalizations made to the results of Section 4. We find that the presence of a nongravitational 4-force (including electromagnetism) does not qualitatively alter the Berezdivin and Sachs result, but merely leads to an additional symmetry constraint on the force field. The restrictive Berezdivin and Sachs result also holds for higher-order field equations. However, the Brans–Dicke theory allows for a less restrictive behavior: the matter symmetry arises from a homothetic symmetry of the metric. The long-range scalar field of the theory inherits this homothetic symmetry. The generalization of the matter symmetry by introducing a conformal matter symmetry is shown to force the spacetime to admit a homothetic metric symmetry. Finally, the attempt to generalize the 2-surface-forming condition to a 3-surface-forming condition is unsuccessful. We are unable to solve the equations that arise from this condition, although we can show that in a special case the spacetime admits a Killing tensor.

In all cases the results and indications point to a surprising "resilience" of geometrical symmetries, in the sense that they are the source for dynamical matter symmetries [a similar point holds for path symmetries (Maartens and Taylor, 1993)]. In the concluding Section 6 we discuss why this may be the case and point to possible generalizations.

2. COLLISION-FREE GAS

We first give a brief review of the relativistic kinetic theory of a collision-free gas [Maartens and Maharaj (1985) and references therein for further details]. The distribution function $f(x, p)$ determines the number of particles near each event x in spacetime M with 4-momenta near p. The momentum space P_x is the region in the tangent space T_xM consisting of future-directed, nonspacelike tangent vectors. Phase space P is then the union of all P_x. The mass-shell $P_x(m)$ consists of all 4-momenta p^a such that $p_a p^a = -m^2$. The mass-shells are the fibers of the phase space $P(m)$ for particles of rest mass m. Then $P(m)$ is a hypersurface in P, which is in turn a region of TM, the tangent bundle. Local coordinates $\{x^a\}$ on M induce local coordinates $\{x^a, p^b\}$ on TM, and $P(m)$ is given locally by

$$g_{ab}(x^c)p^a p^b = -m^2 \tag{1}$$

By choosing p^α ($\alpha = 1, 2, 3$) as coordinates on each mass-shell, we induce local coordinates $\{x^a, p^\alpha\}$ on $P(m)$ with p^0 determined by (1) at each point of $P(m)$. Since free uncharged particles not subject to collisions follow geodesics, all possible uncharged particle motions are given by (1) and

$$\frac{dx^a}{dv} = p^a, \qquad \frac{dp^a}{dv} = -\Gamma^a{}_{bc}p^b p^c \tag{2}$$

where v is an affine parameter; for $m > 0$, $v = (\text{proper time})/m$. The family of intersecting geodesics of M represented by (2) is naturally lifted $[x^a(v) \to (x^a(v), dx^a/dv)]$ into a nonintersecting congruence of phase orbits in P. The tangent vector field to these phase orbits is the Liouville vector field (or geodesic spray)

$$\mathbf{L} = p^a\left(\frac{\partial}{\partial x^a} - \Gamma^b{}_{ca}p^c\frac{\partial}{\partial p^b}\right) \tag{3}$$

From (1) and (3), $\mathbf{L}(m) = 0$, so that $\mathbf{L}$ is tangent to $P(m)$. Furthermore, f is constant along the phase flow since the gas is collision-free. This yields the Liouville (or Vlasov) equation:

$$\mathbf{L}(f) = 0 \tag{4}$$

In the case of charged particles (charge e), the Liouville vector field (3) generalizes to

$$\mathbf{L}_* = \mathbf{L} + eF^a{}_b p^b \frac{\partial}{\partial p^a} \tag{5}$$

where F_{ab} is the electromagnetic field tensor. Equation (5) follows since the integral curves of $\mathbf{L}_*$ are the lifts of charged particle trajectories: $Dp^a/dv = eF^a{}_b p^b$. Then the Liouville equation (4) generalizes to the charged particle case: $\mathbf{L}_*(f) = 0$.

The presence of a velocity-independent, nongravitational 4-force $mh^a(x)$ implies that the particle trajectories are given by $Dp^a/dv = h^a$. Then the natural lifts of these trajectories are the integral curves of the generalized Liouville vector field for motion under a velocity 4-force:

$$\mathbf{L}_* = \mathbf{L} + h^a \frac{\partial}{\partial p^a} \tag{6}$$

In the absence of collisions, the Liouville equation (4) again generalizes to $\mathbf{L}_*(f) = 0$.

The second moment of f defines the kinetic energy-momentum tensor

$$T^{ab} = \int p^a p^b f\, dP \tag{7}$$

where $dP = (-g)^{1/2}\, dp^{0123}$ and the integration is over P_x. For a gas of identical particles (mass m), $dP = (-g)^{1/2} dp^{123}/(-p_0)$ and the integration is over $P_x(m)$. In the case of a self-gravitating gas, the source of the gravitational field is (7). The Einstein field equations

$$G_{ab} \equiv R_{ab} - \tfrac{1}{2}Rg_{ab} = T_{ab} \tag{8}$$

and equation (4) form the self-consistent Einstein–Liouville system of equations, since the integrability conditions

$$T^{ab}{}_{;b} = 0 \tag{9}$$

follow identically from (4) and (7).

The Einstein–Liouville system of equations can be generalized to alternative gravitational field equations. Fourth-order gravitational field equations are derived from a quadratic gravitational Lagrangian (Barrow and Ottewill, 1983):

$$\begin{aligned} G^*_{ab} \equiv R_{ab} - \tfrac{1}{2}Rg_{ab} + 2q[R(R_{ab} - \tfrac{1}{4}Rg_{ab}) - R_{;ab} + g_{ab}\,\Box R] \\ + r[2(R_{abcd} - \tfrac{1}{4}g_{ab}R_{cd})R^{cd} - R_{;ab} + \tfrac{1}{2}g_{ab}\,\Box R + \Box R_{ab}] = T_{ab} \end{aligned} \tag{10}$$

where q and r are constants, $\Box = g^{ab}\nabla_a\nabla_b$, and (9) follows identically from (10), consistently with (4) and (7). The scalar–tensor Brans–Dicke gravitational field equations are (Misner *et al.*, 1993)

$$G^*_{ab} \equiv \phi(R_{ab} - \tfrac{1}{2}Rg_{ab}) - \omega(\phi_{;a}\phi_{;b} - \tfrac{1}{2}g_{ab}\phi_{;c}\phi^{;c})/\phi - (\phi_{;ab} - g_{ab}\,\Box\phi) = T_{ab} \tag{11}$$

where ω is a coupling constant, and (9) implies $\Box\phi = (3+2\omega)^{-1}T^a{}_a$.

3. LIFTED TRANSFORMATIONS ON THE TANGENT BUNDLE AND MATTER SYMMETRIES

In this section we situate the matter symmetries of Berezdivin and Sachs (1973) within a general class of vector fields on TM [see Maartens and Taylor (1993) and the references therein for further details].

The coordinate basis vectors $\{\partial/\partial x^a, \partial/\partial p^b\}$ on TM do not transform covariantly. In order to provide a covariant splitting of vector field components on TM, it is necessary to use the anholonomic connection basis of horizontal and vertical vector fields

$$\mathbf{H}_a = \frac{\partial}{\partial x^a} - \Gamma^b{}_{ac}p^c\frac{\partial}{\partial p^b}, \qquad \mathbf{V}_a = \frac{\partial}{\partial p^a} \tag{12}$$

where

$$[\mathbf{H}_a, \mathbf{H}_b] = -R^d{}_{cab} p^c \mathbf{V}_d, \qquad [\mathbf{H}_a, \mathbf{V}_b] = \Gamma^c{}_{ab} \mathbf{V}_c, \qquad [\mathbf{V}_a, \mathbf{V}_b] = 0 \quad (13)$$

Then any vector field on *TM* can be split covariantly into components with respect to $\{\mathbf{H}_a, \mathbf{V}_b\}$, where the components and the basis vectors transform covariantly (i.e., like rank 1 tensors on *M*). The Liouville vector field (3) can be written as the basis as $\mathbf{L} = p^a \mathbf{H}_a$.

Any vector field Y^a on *M* generates point transformations along its integral curves. In addition, we can define a smooth local rule governing the transport of tangent vectors along the integral curves of Y^a. For a linear transport rule, any tangent vector u^a at x is transported to u'^a at x', where $u'^a = \Omega^a{}_b(x; c)u^b$. The linear *transport lifts* are the vector fields on *TM* which define these transformations (Maartens and Taylor, 1993):

$$\mathbf{Y}^{\mathbf{A}} = Y^a(x)\mathbf{H}_a + A^a{}_b(x) p^b \mathbf{V}_a \qquad (14a)$$

where

$$A^a{}_b(x) = \partial \Omega^a{}_b(x; 0)/\partial \epsilon + \Gamma^a{}_{bc}(x) Y^c(x) \qquad (14b)$$

is the rank 2 tensor field on *M* which covariantly defines the transport of tangent vectors along Y^a. Suitable choices of $A^a{}_b$ allow us to regain all of the standard lifted vector fields as special cases of (14) (Maartens and Taylor, 1993). In particular, the matter symmetries of Berezdivin and Sachs are members of the subclass of (14) in which the transport rule Ω is *Lorentz transport* along Y^a. This implies

$$A_{(ab)} = 0 \qquad (15a)$$

The matter symmetry vector fields in addition leave the distribution function invariant:

$$\mathbf{Y}^{\mathbf{A}}(f) = 0 \qquad (15b)$$

Note that (15a) implies $\mathbf{Y}^{\mathbf{A}}(m) = 0$. Thus by (1), (3), (4), and (15) the vector fields $\mathbf{L}$ and $\mathbf{Y}^{\mathbf{A}}$ are everywhere tangent to the 6-dimensional hypersurface $\{m = \text{const}, f = \text{const}\}$ of *P*. Our definition is equivalent to the original definition of Berezdivin and Sachs: *a matter symmetry arises when observers at different points along Y^a curves, using Lorentz frames, measure the distribution to be the same.*

Matter symmetries in fact arise in a well-known class of distributions—those which are isotropic in momentum space relative to some 4-velocity field u^a:

$$f(x, p) = F(x, u_a p^a) \qquad (16)$$

Clearly f is invariant under the isotropy group of u^a in momentum space, i.e., under the rotation subgroup of the Lorentz group, with Lie algebra generators of the form $A^a{}_b p^b \partial / \partial p^a$, where $A_{ab} u^b = 0 = A_{(ab)}$. Thus the generators are matter symmetry vector fields that are vertical ($Y^a = 0$) (Berezdivin and Sachs, 1973). Ehlers *et al.* (1968) showed that for a dynamically isotropic distribution of the form (16) the spacetime is either stationary or Robertson–Walker. In this case, matter symmetries give rise to very restrictive geometrical symmetries.

An important special case of linear transport lift arises when the transport rule is Lie transport along Y^a, so that $A_{ab} = Y_{a;b}$ and we write the *complete* (*or Lie*) *lift as*

$$\tilde{\mathbf{Y}} = Y^a \mathbf{H}_a + Y^a{}_{;b} p^b \mathbf{V}_a \tag{17}$$

The rate of change of the distribution function under a spacetime symmetry Y^a is then $\tilde{\mathbf{Y}}(f)$ (Maartens and Maharaj, 1985; Ehlers, 1971). By (15a), for a complete lift (17) to be a matter symmetry, Y^a must be a Killing vector field.

An alternative approach to dynamical symmetries is via invariance of the phase orbits rather than invariance of the distribution function as in (15b). A vector field $\mathbf{Z}$ on phase space is a *dynamical path symmetry* of $\mathbf{L}$ if it maps curves of $\mathbf{L}$ (phase orbits) into each other, i.e. (Maartens and Taylor, 1993)

$$[\mathbf{Z}, \mathbf{L}] = l\mathbf{L} \tag{18}$$

for some scalar $l(x, p)$. If $l = 0$, then $\mathbf{Z}$ is known as a dynamical Lie symmetry (in this case, the parameter is invariant under the mapping of the phase orbit). *A matter symmetry that is also a path symmetry is the complete lift of a homothetic Killing vector field* (independently of any field equations (Maartens and Taylor, 1993).

The identity (Ehlers, 1971)

$$\Omega_c \int p^a \cdots p^b \Psi(f)\, dP = \int p^a \cdots p^b \Psi'(f) \mathbf{H}_c(f)\, dP$$

leads, for $\Psi(f) = f$, to

$$T^{ab}{}_{;c} = \int p^a p^b \mathbf{H}_c(f)\, dP$$

which gives, using integration by parts,

$$\int p^a p^b \mathbf{Y}^{\mathrm{A}}(f)\, dP = T^{ab}{}_{;c} Y^c - A^a{}_c T^{cb} - T^{ac} A^b{}_c - A^c{}_c T^{ab} \tag{19}$$

Equation (19) holds for any linear transport lift of the form (14a). If $\mathbf{Y}^{\mathbf{A}}$ is a matter symmetry, then (15) and (16) imply

$$\mathscr{L}_Y T^{ab} = (A^a{}_c - Y^a{}_{;c})T^{cb} + T^{ac}(A^b{}_c - Y^b{}_{;c}) \tag{20}$$

which was given by Berezdivin and Sachs (1973). Using the field equations for T^{ab}, (20) determines the fundamental link between dynamical matter symmetries and the geometrical properties of spacetime. In fact, Berezdivin and Sachs did not use equation (20). In this paper (20) is the crucial equation, and we are able to simplify considerably the derivation of their result by using it.

It is not surprising, but nontrivial to show, that matter symmetries form a Lie algebra. Using (13)–(15), we find that for constants s and t

$$s\mathbf{Y}^{\mathbf{A}} + t\mathbf{Z}^{\mathbf{B}} = \mathbf{W}^{\mathbf{C}}$$

where $W = sY + tZ$ and $C = sA + tB$, which clearly implies $C_{(ab)} = 0$. Also (Maartens and Taylor, 1993)

$$[\mathbf{Y}^{\mathbf{A}}, \mathbf{Z}^{\mathbf{B}}] = [\mathbf{Y}, \mathbf{Z}]^{\mathbf{C}}$$

where

$$C = \nabla_Y B - \nabla_Z A - [A, B] - R(Y, Z)$$

is a rank 2 tensor field on M and $R(Y, Z)_{ab} = R_{abcd} Y^c Z^d$. By the symmetries of R_{abcd} and the skewness of A and B, it follows that C is skew and consequently that $[\mathbf{Y}, \mathbf{Z}]^{\mathbf{C}}$ is a matter symmetry.

Berezdivin and Sachs (1973) pointed out that if $\mathbf{Y}^{\mathbf{A}}$ is a matter symmetry, then any scaling that is constant on momentum space

$$\mathbf{Y}^{\mathbf{A}} \to \bar{\mathbf{Y}}^{\bar{\mathbf{A}}} = e^{\lambda(x)}\mathbf{Y}^{\mathbf{A}} \tag{21a}$$

preserves its properties (15) as a matter symmetry, so that only the directions of Y^a and A_{ab} on M are important and not their magnitudes. We note also that scaling preserves the Lie algebra since it is linear and since

$$[e^{\lambda(x)}\mathbf{Y}^{\mathbf{A}}, e^{\phi(x)}\mathbf{Z}^{\mathbf{B}}] = e^{\lambda(x)+\phi(x)}\{[\mathbf{Y}, \mathbf{Z}]^{\mathbf{C}} + (\mathscr{L}_Y \phi)\mathbf{Z}^{\mathbf{B}} - (\mathscr{L}_Z \lambda)\mathbf{Y}^{\mathbf{A}}\}$$

(Thus the Lie algebra of matter symmetries is infinite-dimensional over the phase space, but not on each momentum space.) We will refer to (21a) as a gauge transformation, so that the basic properties of matter symmetries are gauge invariant. As expected, the key equation (20) is gauge invariant. Equation (21a) gives rise to an important gauge freedom:

$$\bar{Y}_{a;b} = e^{\lambda}[Y_{a;b} + Y_a \lambda_{;b}] \tag{21b}$$

In particular, if $Y_{a;b} = Y_a \sigma_b$, where $\sigma_{[a;b]} = 0$, then by (21b), Y_a may be rescaled to a Killing vector field.

4. THE BEREZDIVIN AND SACHS RESULT

In deriving their main result Berezdivin and Sachs do not motivate the 2-surface-forming condition. It is diffiicult to see any alternative approach in searching for conditions implied by the matter symmetry properties. However, it is not clear what physical meaning may be attached to this assumption. Geometrically, the assumption is a natural generalization of a dynamical path symmetry: the phase orbits are mapped by $\mathbf{Y^A}$ into paths within the $\mathbf{Y^A}$, $\mathbf{L}$ 2-surfaces, rather than into each other (i.e., into the $\mathbf{L}$ 1-surfaces).

At each point in phase space, $\mathbf{Y^A}$ and $\mathbf{L}$ span a 2-plane in the tangent space. However, in general the tangent 2-planes may not mesh together to form 2-surfaces. By Frobenius' theorem the condition for $\mathbf{Y^A}$ and $\mathbf{L}$ to be 2-surface-forming is that $[\mathbf{Y^A}, \mathbf{L}]$ be in the tangent 2-plane at each point. If this condition does not hold, $[\mathbf{Y^A}, \mathbf{L}]$ together with $\mathbf{Y^A}$ and $\mathbf{L}$ generates 3-planes, which, in turn, may or may not mesh together to form 3-surfaces. The most general case is that the matter symmetry $\mathbf{Y^A}$ and dynamical vector field $\mathbf{L}$ generate 6-dimensional integral surfaces. [Berezdivin and Sachs (1973) claim that the most general case is 8-dimensional; however, the constraint equations $\mathbf{L}(m) = 0 = \mathbf{L}(f)$, $\mathbf{Y^A}(m) = 0 = \mathbf{Y^A}(f)$ restrict the integral curves of $\mathbf{L}$ and $\mathbf{Y^A}$ (and all their Lie brackets) to lie in the intersection of the hypersurfaces $\{m = \text{const}\}$ and $\{f = \text{const}\}$, which is of dimension 6]. Berezdivin and Sachs were unable to make progress with the general case, and assumed that $\mathbf{Y^A}$ is a 2-surface-forming with $\mathbf{L}$. Using this assumption, they proved the following result (Berezdivin and Sachs, 1973).

For a gas obeying the Einstein–Liouville equations, a correctly scaled matter symmetry that is 2-surface-forming with the Liouville vector field is the complete lift of a Killing vector field on spacetime.

In this section we will outline an improved derivation of their result, which, unlike their derivation, may be applied to the case when $\mathbf{Y^A}$ is not 2-surface forming with $\mathbf{L}$ (see Section 5). The 2-surface-forming condition is

$$[\mathbf{Y^A}, \mathbf{L}] = k(x, p)\mathbf{Y^A} + l(x, p)\mathbf{L} \tag{22}$$

for some scalars k and l on P. By (18), Y^A is a dynamical path symmetry if $k = 0$. By (3) and (12)–(15)

$$[\mathbf{Y^A}, \mathbf{L}] = (A^a{}_b - Y^a{}_{;b})p^b\mathbf{H}_a + (R^a{}_{bcd}Y^d - A^a{}_{b;c})p^b p^c\mathbf{V}_a \tag{23}$$

By (21) and (22), the 2-surface property is gauge invariant, with gauge freedom

$$\bar{k} = k - \lambda_{;a}p^a, \qquad \bar{l} = e^{\lambda}l \tag{24}$$

Thus there is a gauge freedom to scale away k provided it is of the form $k(x,p)=\alpha_a(x)p^a$, where $\alpha_{[a;b]}=0$. (This will be shown to be the case below.) Then in this case $\bar{\mathbf{Y}}^{\mathrm{A}}$ is a dynamical path symmetry. (Clearly the dynamical path symmetry property is not invariant under the matter symmetry gauge transformations.)

Expanding k and l on each momentum space

$$k(x,p)=\alpha(x)+\alpha_a(x)p^a+\alpha_{ab}(x)p^ap^b+\cdots \tag{25a}$$

$$l(x,p)=\beta(x)+\beta_a(x)p^a+\beta_{ab}(x)p^ap^b+\cdots \tag{25b}$$

we note that the gauge freedom (24) implies $\alpha\to\alpha-\lambda_{;a}p^a$, $\beta_{a_1\cdots a_r}\to e^{\lambda}\beta_{a_1\cdots a_r}$ $(r=0,1,\ldots)$. Substituting (25) into (22) and comparing with (23) yields a system of equations in powers of p^a. To fourth order these give

$$\alpha Y_a=0=\alpha A_{ab} \tag{26a}$$

$$A_{ab}-Y_{a;b}=Y_a\alpha_b+Bg_{ab} \tag{26b}$$

$$R_{a(bc)d}Y^d-A_{a(b;c)}=A_{a(b}\alpha_{c)} \tag{26c}$$

$$Y_a\alpha_{(bc)}+g_{a(b}\beta_{c)}=0=Y_a\alpha_{(bcd)}+g_{a(b}\beta_{cd)} \tag{26d}$$

$$A_{a(b}\alpha_{cd)}=0=A_{a(b}\alpha_{cde)} \tag{26e}$$

The higher-order equations have the same form as (26a), (26d), and (26e). Then it is easily shown that the only possible nonzero coefficients are α_a and β, so that $k(x,p)=\alpha_a(x)p^a$ and $l(x,p)=\beta(x)$. Covariantly differentiating (26b) and then symmetrizing on ab and antisymmetrizing on bc yields (Berezdivin and Sachs, 1973)

$$g_{ab}\beta_{;c}-\beta_{;(a}g_{b)c}-\beta[\alpha_{(a}g_{b)c}-g_{ab}\alpha_c]+[Y_a\alpha_{[b;c]}+Y_b\alpha_{[a;c]})=0 \tag{27}$$

Assuming $\beta\neq 0$, we can choose the gauge potential in (21) and (24) to be $\lambda=-\log\beta$, so that (27) reduces to

$$g_{ab}\bar{\alpha}_c-\bar{\alpha}_{(a}g_{b)c}+(\bar{Y}_a\bar{\alpha}_{[b;c]}+\bar{Y}_b\bar{\alpha}_{[a;c]})=0 \tag{28}$$

Following Berezdivin and Sachs (1973), we contract (28) with $\xi^a\xi^b$, where ξ^a is arbitrary subject to $\xi^a\xi_a\neq 0$, $\xi^a\bar{Y}_a=0=\xi^a\bar{\alpha}_a$. Then (28) yields $\bar{\alpha}_a=0$, so that $\bar{\mathbf{Y}}^{\mathrm{A}}$ is a dynamical path symmetry by (18). Thus if $\beta\neq 0$, the 2-surface-forming assumption reduces to the assumption that the matter symmetry is a dynamical path symmetry—and therefore it is the complete lift of a homothetic Killing vector (Maartens and Taylor, 1993). Indeed, by (26b) $(\beta\neq 0)$

$$\bar{Y}_{a;b}=\bar{A}_{ab}-g_{ab} \quad\Rightarrow\quad \mathscr{L}_{\bar{Y}}g_{ab}=-2g_{ab} \tag{29a}$$

At this point Berezdivin and Sachs employed an involved and ingenious argument to show that (29a) produces a contradiction when the field equations (8) are invoked. However, using (20), we can bypass their argument to immediately arrive at the result. Equation (20) with (29a) gives

$$\mathscr{L}_{\bar{Y}} T^{ab} = 2T^{ab} \tag{29b}$$

However, Einstein's field equations (8) with (29a) imply

$$\mathscr{L}_{\bar{Y}} T^{ab} = 4T^{ab} \tag{30}$$

using the identities for homothetic Lie derivatives (Maartens *et al.*, 1986). This contradiction means $\beta = 0$, and (29a) is not true. (Note that a nonzero cosmological constant does not affect this result.) The importance of (20) is evident when comparing the complicated arguments of Berezdivin and Sachs with the straightforward derivation above. With $\beta = 0$, (27) implies $\alpha_{[a;b]} = 0$, so that (locally) $\alpha_a = \lambda_{;a}$ for some λ. If we take λ as gauge potential, (21) and (26b) show that $\bar{Y}_{a;b} = \bar{A}_{ab}$, so that $\bar{Y}$ is Killing and $\bar{\mathbf{Y}}^{\mathrm{A}}$ is its complete lift. We note that since $\bar{k} = 0 = \bar{l}$, $\bar{\mathbf{Y}}^{\mathrm{A}}$ is not only a dynamical path symmetry of **L**, but furthermore a Lie symmetry (Maartens and Taylor, 1993). Thus it turns out that the 2-surface assumption reduces to the condition that the matter symmetry is also a Lie symmetry of **L**.

Berezdivin and Sachs have then shown that, at least in the 2-surface-forming case, the Einstein field equations reduce the dynamical symmetry to a geometrical symmetry. In Section 5 we will investigate whether we can avoid this restrictive result by considering nongravitational forces, by looking at alternative field equations, or by relaxing the 2-surface-forming condition.

5. EXTENSIONS OF THE BEREZDIVIN AND SACHS RESULT

5.1. Gas Particles Subject to Nongravitational Forces

We follow Berezdivin and Sachs (1973) in assuming that $\mathbf{Y}^{\mathrm{A}}$ is 2-surface-forming with **L**, but we consider whether nongravitational forces may alter the result that the dynamical matter symmetry reduces to a Killing symmetry. We thus investigate the possibility that the dynamical symmetry degenerates to a geometrical symmetry mainly because the particles are in free fall. In fact this is not the case; an external force does not qualitatively alter the conclusion of Section 4.

First consider the case of charged particles, where the 4-force is the Lorentz force on each particle due to the collectively generated electromagnetic field. Using the 2-surface-forming condition (22), with the generalized

Liouville vector field (5) and the definition (14) of the matter symmetry, we get the charged particle generalization of (23),

$$
\begin{aligned}
[\mathbf{Y}^{\mathbf{A}}, \mathbf{L}_*] &= k(x,p)\mathbf{Y}^{\mathbf{A}} + l(x,p)\mathbf{L}_* \\
&= (A^a{}_b - Y^a{}_{;b})p^b\mathbf{H}_a \\
&\quad + \{e(F^a{}_{b;c}Y^c + [F,A]^a{}_b)p^b + (R^a{}_{bcd}Y^d - A^a{}_{b;c})p^bp^c\}\mathbf{V}_a \qquad (31)
\end{aligned}
$$

which yields two conditions:

$$(A^a{}_b - Y^a{}_{;b})p^b = kY^a + lp^a \qquad (32a)$$

$$e(F^a{}_{b;c}Y^c + [F,A]^a{}_b)p^b + (R^a{}_{bcd}Y^d - A^a{}_{b;c})p^bp^c = kA^a{}_bp^b + lF^a{}_bp^b \qquad (32b)$$

Equation (32a) is identical to the equation for the neutral ($e=0$) case in Section 4, and using the expansions (25) again leads to $k(x,p) = \alpha_a(x)p^a$, $l(x,p) = \beta(x)$. An argument identical to that of Section 4 shows that $\beta = 0$, $\alpha_a = \lambda_{;a}$, and $\bar{\mathbf{Y}}^{\mathbf{A}}$ is the complete lift of the Killing vector field $\bar{Y}^a$. Then (32b) gives the additional symmetry condition that the electromagnetic field be invariant under the Killing symmetry:

$$\mathscr{L}_{\bar{Y}}F_{ab} = 0$$

Thus the electromagnetic force fails to prevent the dynamical matter symmetry from reducing to a Killing symmetry. On the contrary, the dynamical matter symmetry forces the electromagnetic field to obey the same geometrical symmetry as the spacetime metric.

The same conclusion emerges when we look at a velocity-independent nongravitational 4-force mn^a. The 2-surface-forming condition with the generalized Liouville vector field (6) is

$$
\begin{aligned}
[\mathbf{Y}^{\mathbf{A}}, \mathbf{L}_*] &= k(x,p)\mathbf{Y}^{\mathbf{A}} + l(x,p)\mathbf{L}_* \\
&= (A^a{}_b - Y^a{}_{;b})p^b\mathbf{H}_a \\
&\quad + \{(h^a{}_{;b}Y^b - A^a{}_bh^b) + (R^a{}_{bcd}Y^d - A^a{}_{b;c})p^bp^c\}\mathbf{V}_a \qquad (33)
\end{aligned}
$$

Equation (33) yields (32a) and the additional condition

$$(h^a{}_{;b}Y^b - A^a{}_bh^b) + (R^a{}_{bcd}Y^d - A^a{}_{b;c})p^bp^c = kA^a{}_bp^b + lh^a \qquad (34)$$

As before, (32a) leads to a Killing symmetry. Then (34) implies the same geometrical symmetry for the 4-force as for the spacetime metric:

$$\mathscr{L}_{\bar{Y}}h^a = 0$$

In summary: *For a gas subject to a Lorentz or velocity-independent 4-force and obeying the Einstein–Liouville equations, a matter symmetry that is 2-surface-forming with the Liouville vector field reduces to a Killing symmetry.*

5.2. Alternative Gravitational Field Equations

The derivation of the results of Section 4 is dependent on the form of the field equations. We investigate the effect that alternative gravitational field equations have on this result. We find that the result is unchanged in the case of the higher-order field equations (10). However, the Brans–Dicke field equations (11) do not contradict the matter symmetry condition (20) in the event of a $\bar{Y}_a$ being a homothetic Killing vector, provided the long-range scalar field ϕ is self-similar.

In Section 4, without using the field equations but assuming $\beta \neq 0$, we reduced the 2-surface-forming condition to a single equation, (29a), with $\bar{Y}_a$ a homothetic Killing vector, and we used the key equation (20) to derive (29b). We now show that the higher-order field equations are incompatible with (29). Using the general results for homothetic Lie derivatives (Maartens *et al.*, 1986) and the field equations (10), we get

$$\mathscr{L}_{\bar{Y}} G^*_{ab} = 2(G^*_{ab} - G_{ab}) \tag{35}$$

Then (10), (29b), and (35) imply $G_{ab} = 2T_{ab}$. Taking the Lie derivative of this, we get $\mathscr{L}_{\bar{Y}} T_{ab} = 0$, which contradicts (29b). (Note the surprising result that these equations following from (35) have no explicit dependence on the coupling constants q and r of (10).) Once again we conclude that $\beta = 0$.

In summary: *For a gas obeying the Liouville and higher-order field equations, a matter symmetry that is 2-surface-forming with the Liouville vector field reduces to a Killing symmetry.*

A qualitatively different result arises from the Brans–Dicke field equations (11). Using the general results for homothetic Lie derivatives, (11), and (29b), we find

$$\begin{aligned}(1 + \phi'/2\phi)T_{ab} = &-\omega\phi'(\phi_a\phi_b - \tfrac{1}{2}g_{ab}\phi^c\phi_c)/\phi^2 \\ &+ \omega(\phi'_a\phi_b + \phi_a\phi'_b - g_{ab}\phi'_c\phi^c)/2\phi \\ &- \phi'(\phi_{a;b} - g_{ab}\Box\phi)/2\phi + (\phi'_{a;b} - g_{ab}\Box\phi')/2\end{aligned} \tag{36}$$

where $\phi_a = \phi_{;a}$ and $\phi' = \mathscr{L}_{\bar{Y}}\phi$. From the trace of (36) we see that self-similarity of the long-range scalar field

$$\mathscr{L}_{\bar{Y}}\phi = -2\phi \tag{37}$$

implies that (36) is identically satisfied. [Note that (37) not only echoes the homothetic symmetry of the metric tensor (40); but is also the necessary condition for $\mathscr{L}_{\bar{Y}}(\Box\phi) = 0$.] Thus $\beta \neq 0$ is not ruled out, and $\bar{\mathbf{Y}}^{\mathbf{A}}$ is a dynamical path symmetry (not in general a Lie symmetry) that is the complete lift of a homothetic Killing vector. The Brans–Dicke field equations (11) and the matter symmetry conditions (29) are thus compatible, provided (37) holds. Consequently the result of Section 4 is relaxed to:

For a gas obeying the Liouville and Brans–Dicke field equations, a matter symmetry that is 2-surface-forming with the Liouville vector field reduces to a homothetic Killing symmetry provided the long-range scalar field transforms homothetically.

5.3. Conformal Matter Symmetries

It may be that the Berezdivin and Sachs invariance (15b) is too restrictive, and that a more genuinely dynamical symmetry will result from generalizing (15b). We try a conformal generalization:

$$\mathbf{Y}^{\mathbf{A}}(f) = -2\psi(x)f \tag{38}$$

where $A_{(ab)} = 0$. [Note that (38) was also suggested by Oliver and Davis, (1979) in another context.] We retain the 2-surface-forming assumption (22) and derive (29) as before. Substituting (38) into the identity (19) gives

$$\mathscr{L}_Y T^{ab} - (A^a{}_c - Y^a{}_{;c})T^{cb} - T^{ac}(A^b{}_c - Y^b{}_{;c}) = -2\psi T^{ab} \tag{39}$$

By (29) $\bar{Y}^a$ is a homothetic Killing vector field. The gauge invariant (39) gives, with (29a),

$$\mathscr{L}_{\bar{Y}} T^{ab} = (2 - 2\bar{\psi})T^{ab} \tag{40}$$

where $\bar{\psi} = \psi/\beta$. Comparing (30) and (40), we see that $\psi = -\beta$. Thus we are not forced to conclude that $\beta = 0$ as was the case for a nonconformal matter symmetry.

In summary: *For a gas obeying the Einstein–Liouville equations, a conformal matter symmetry that is 2-surface-forming with the Liouville vector field arises from a homothetic symmetry on spacetime.*

It may be possible to generalize (15b) in other, more dynamical, directions. Our conformal generalization allows only a slight modification of the restrictive Berezdivin and Sachs result from a Killing to a homothetic symmetry.

5.4. Matter Symmetries That Are Not 2-Surface-Forming with *L*

Our final attempted generalization is the most difficult and the least conclusive. In the case when $\mathbf{Y}^{\mathbf{A}}$ and $\mathbf{L}$ generate 3-surfaces, we are unable to determine whether in general the matter symmetry collapses to a geometrical symmetry. However, under special conditions we find that $Y_{(a;b)}$ is a Killing tensor.

By (29)

$$\mathbf{W} \equiv [\mathbf{Y}^{\mathbf{A}}, \mathbf{L}) = (A^a{}_b - Y^a{}_{;b})p^b\mathbf{H}_a + (R^a{}_{bcd}Y^d - A^a{}_{b;c})p^bp^c\mathbf{V}_a \tag{41}$$

and the conditions that $\mathbf{Y}^{\mathrm{A}}$ and $\mathbf{L}$ generate 3-dimensional integral surfaces are

$$[\mathbf{L}, \mathbf{W}] = k(x,p)\mathbf{Y}^{\mathrm{A}} + l(x,p)\mathbf{L} + n(x,p)\mathbf{W} \tag{42a}$$

$$[\mathbf{Y}^{\mathrm{A}}, \mathbf{W}] = k'(x,p)\mathbf{Y}^{\mathrm{A}} + l'(x,p)\mathbf{L} + n'(x,p)\mathbf{W} \tag{42b}$$

for some scalars k, k', l, l', n, n' on P. Note that the 3-surface property is gauge invariant: the gauge freedom (21) gives

$$\bar{\mathbf{W}} = e^{\lambda}[\mathbf{W} - (\lambda_{;a}p^a)\mathbf{Y}^{\mathrm{A}}] \tag{43a}$$

$$\bar{k} = k - \lambda_{;ab}p^a p^b + (\lambda_{;a}p^a)^2 + n(\lambda_{;a}p^a), \qquad \bar{l} = e^{\lambda}l, \qquad \bar{n} = n + 2\lambda_{;a}p^a \tag{43b}$$

and more complicated gauge transformations for k', l', and n', which we will not use.

Now by (3), (14a), and (41)

$$\begin{aligned}[\mathbf{L}, \mathbf{W}] = {} & 2A^a{}_{b;c} - Y^a{}_{;bc} - R^a{}_{bcd}Y^d)p^b p^c \mathbf{H}_a \\ & + (R^a{}_{bce;d}Y^e + 2R^a{}_{bce}Y^e{}_{;d} - R^a{}_{bce}A^e{}_d - A^a{}_{b;cd})p^b p^e p^d \mathbf{V}_a \end{aligned} \tag{44}$$

$$\begin{aligned}[\mathbf{Y}^{\mathrm{A}}, \mathbf{W}] = {} & (A^a{}_{b;c}Y^c - Y^a{}_{;bc}Y^c + Y^a{}_{;c}Y^c{}_{;b} - 2Y^a{}_{;c}A^c{}_b + A^a{}_c A^c{}_b)p^b \mathbf{H}_a \\ & + \{[R^a{}_{bce;d}Y^e + R^a{}_{bce}Y^e{}_{;d} - A^a{}_{bc;d}]Y^d \\ & + 2R^a{}_{bde}Y^e A^d{}_c - R^a{}_{bde}Y^e A^d{}_{;c} - R^a{}_{bde}Y^e Y^d{}_{;c} - A^a{}_{b;d}A^d{}_c \\ & + A^a{}_{b;d}Y^d{}_{;c} + R^a{}_{dce}Y^e A^d{}_b - R^d{}_{bce}Y^e A^a{}_d\}p^b p^c \mathbf{V}_a \end{aligned} \tag{45}$$

Owing to the complexity of the right-hand sides of (44) and (45), the method of Berezdivin and Sachs (1973) is not applicable. Our method of Section 4 carries over to this case; we expand k, l, and n in each momentum space to get (25) and

$$n(x,p) = \gamma(x) + \gamma_a(x)p^a + \gamma_{ab}(x)p^a p^b + \cdots \tag{46}$$

The functions k', l', and n' in (42b) can be expanded in a similar way. Substituting (25), (46), and their primed counterparts into (42), and using (44) and (45), we obtain, after lengthy calculations, a system of equations in powers of p^a. To third order, these give

$$\alpha Y_a = 0 = \alpha A_{ab} \tag{47a}$$

$$Y_a \alpha_b + \beta g_{ab} + \gamma(A_{ab} - Y_{a;b}) = 0 \tag{47b}$$

$$Y_a \alpha_{(bc)} + g_{a(b}\beta_{c)} + A_{a(b}\gamma_{c)} - Y_{a;(b}\gamma_{c)} = 2A_{a(b;c)} - Y_{a;(bc)} - R_{a(bc)d}Y^d \tag{47c}$$

$$A_{a(b}\alpha_{c)} + \gamma[R_{a(bc)d}Y^d - A_{a(b;c)}] = 0 \tag{47d}$$

$$A_{a(b}\alpha_{cd)} + R_{a(bc|e|}Y^e\gamma_{d)} - A_{a(b;c}\gamma_{d)} = R_{a(bc|e|;d)}Y^e + 2R_{a(bc|e|}Y^e{}_{;d)} - A_{a(b;cd)} - R_{a(bc|e|}A^e{}_{d)} \quad (47e)$$

$$\alpha' Y_a = 0 = \alpha' A_{ab} \quad (48a)$$

$$Y_a\alpha'_b + \beta' g_{ab} + \gamma'(A_{ab} - Y_{a;b}) = A_{ac}A^c{}_b + A_{ab;c}Y^c - Y_{a;bc}Y^c + Y_{a;c}Y^c{}_{;b} - 2Y_{a;c}A^c{}_b \quad (48b)$$

$$Y_a\alpha'_{(bc)} + g_{a(b}\beta'_{c)} + A_{a(b}\gamma'_{c)} - Y_{a;(b}\gamma'_{c)} = 0 \quad (48c)$$

$$A_{a(b}\alpha'_{c)} + \gamma'[R_{a(bc)d}Y^d - A_{a(b;c)}] = [R_{a(b)e;d}Y^e + R_{a(bc)e}Y^e{}_{;d} - A_{a(b;c)d}]Y^d + R_{a(b|de|}Y^eA^d{}_{c)} - R_{a(b|de|}Y^eY^d{}_{;c)} - A_{a(b;|d|}A^d{}_{c)} - A_{a(b;|d|}Y^d{}_{;c)} + R_{a(b|de|}Y^dA^e{}_{c)} + R_{ad(b|e|}Y^eA^d{}_{c)} - A_{ad}R^d_{(bc)e}Y^e \quad (48d)$$

$$Y_a\alpha'_{(bcd)} + g_{a(b}\beta'_{cd)} + A_{a(b}\gamma'_{cd)} - Y_{a;(b}\gamma'_{cd)} = 0 \quad (48e)$$

$$A_{a(b}\alpha'_{cd)} + R_{a(bc|e|}Y^e\gamma'_{d)} - A_{a(b;c}\gamma'_{d)} = 0 \quad (48f)$$

We see that equations (47b) and (47d) are related to the 2-surface-forming conditions (26a) and (26b) according to the nature of the coefficient γ. If $\gamma \neq 0$, we may use a rescaling with $\lambda = \log\gamma$ to transform (47b) and (47d) into (26a) and (26b). In this instance, the 3-surface case degenerated to the 2-surface case. In order to have a nondegenerate 3-surface case we require that $\gamma = 0$, which implies $\beta = 0 = \alpha_a$ by (47b) and (47d). (Note that $\gamma = 0 = \beta = \alpha_a$ is gauge invariant.) Furthermore, either (26a) or (26b) does not hold. If we define

$$U_{ab} = Y_{a;b} - A_{ab} \quad (49a)$$

$$V^a_{bc} = \mathscr{L}_Y\Gamma^a{}_{bc} - U^a{}_{(b;c)} \quad (49b)$$

then for either $U_{ab} \neq 0$ or $V_{abc} \neq 0$ we have a nondegenerate 3-surface case for which $[\mathbf{Y}^A, \mathbf{L}] \notin \mathrm{span}\,\{\mathbf{Y}^A, \mathbf{L}\}$. The definitions (49) show that in principle Y_a (or $\bar{Y}_a$) may be Killing with $U_{ab} \neq 0 \neq V_{abc}$. However, we were unable to prove that this is true in general, or indeed to make further progress in the general case.

We now examine the special subcase $k = l = n = 0$ of $\gamma = 0$. Although $\mathbf{L}$ and $\mathbf{W}$ are now 2-surface-forming and commuting, $\mathbf{Y}^A$ is not necessarily 2-surface-forming with $\mathbf{L}$. [Note that the gauge freedom (43b) only preserves this subcase if λ is constant.] Then (47c) and (47e) reduce to

$$2A_{a(b;c)} - Y_{a;(bc)} - R_{a(bc)d}Y^d = 0 \qquad (50a)$$

$$R_{a(bc|e|;d)}Y^e + 2R_{a(bc|e|}Y^e{}_{;d)} - R_{a(bc|e|}A^e{}_{d)} - A_{a(b;cd)} = 0 \qquad (50b)$$

Equation (50a) implies that $Y_{(a;bc)} = 0$, which means that $Y_{(a;b)}$ is a Killing tensor. By (49) and (50)

$$V^a{}_{bc} = U^a{}_{(b;c)} = \tfrac{1}{2}\mathscr{L}_Y \Gamma^a{}_{bc}, \qquad U_{a(b;cd)} = R_{a(bc|e|}U^e{}_{d)} \qquad (51)$$

Even in this special subcase we are unable to determine the general solution via (51) or otherwise. We would like to find an example where the matter symmetry does not reduce to a Killing symmetry. The integrability conditions (51) are identically satisfied if Y_a is homothetic (i.e., $U_{(ab)} = \psi g_{ab}$, $\psi_{;a} = 0$). However, it is not clear to us whether the remaining equations in (48), or the higher-order counterparts of (47) and (48), will force Y_a to be Killing. Note that we have not used any field equtions.

In summary: *We are unable to extend the Berezdivin and Sachs result to the 3-surface case—and clearly the more general cases will be yet more complex. In the 3-surface subcase where* **L** *commutes with* $[\mathbf{Y}^A, \mathbf{L}]$, $Y_{(a;b)}$ *is a Killing tensor, regardless of the gravitational field equations.*

There may be another approach which bypasses the surface-forming conditions and allows for a definite answer. The indications from the 3-surface-forming equations are that the matter symmetry is still likely to reduce to a geometrical symmetry, even if not necessarily an isometry.

6. CONCLUDING REMARKS

In the case of a macroscopic fluid model, Bonnor and Ellis (1986) introduce an observationally and thermodynamically based definition of homogeneity—but this dynamical homogeneity does *not* lead in general to a geometrical homogeneity. In the microscopic kinetic model, the dynamical matter symmetry appears to lead inevitably to a geometrical symmetry. At first sight, one may suspect that these contrasting results arise from the fact that the microscopic dynamical symmetry is too detailed and stringent, whereas the macroscopic dynamical symmetry allows for more latitude.

However, the problem goes deeper than this first impression. The fact is that the dynamical matter symmetries reduce to geometrical symmetries only when we impose *additional assumptions* about their surface-forming properties. Without these additional assumptions, it is unclear what happens. Even our attempt to relax the 2-surface-forming assumption involves a 3-surface-forming assumption.

These surface-forming assumptions are not merely technical. As we pointed out in the 2-surface case, the assumption amounts to *assuming that the dynamical matter symmetry is simultaneously a dynamical path symmetry* (or a modified type of path symmetry). Thus in fact we remain unclear about the *essential* nature of matter symmetries—because the "pure" matter symmetry case appears intractable without simplifying assumptions.

Path and matter symmetries are very different approaches to dynamical symmetries in kinetic theory. The path symmetry has no apparent observational basis and leads directly to a geometrical symmetry (at least in the linear case) (Maartens and Taylor, 1993). It therefore seems to us to be a somewhat unsatisfactory concept of dynamical symmetry. In contrast, the matter symmetry of Berezdivin and Sachs is observationally based—but appears to be too broad without further assumptions.

The implications of a "pure" matter symmetry remain unknown. There may be another approach, alternative to our simplification and generalization of the Berezdivin and Sachs approach, which uncovers the consequences of a matter symmetry without surface-forming assumptions. Alternatively, there may be additional dynamical (as opposed to phase-space geometrical) aspects which could naturally be added to the matter symmetry definition or used to modify it and which would lead to answers about the nature of genuinely dynamical symmetries in kinetic theory. One possible approach may be to seek a kinetic foundation for the fluid postulate of uniform thermal histories of Bonnor and Ellis (1986). These issues are currently under investigation.

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